Hybrid vs. nonsmooth dynamical systems Why can it be so difficult to model physics in hybrid modelers? [1]

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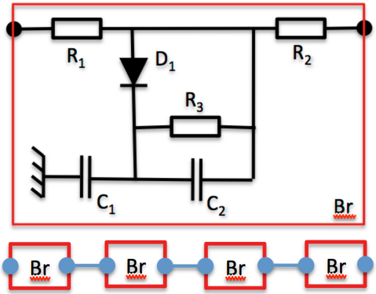
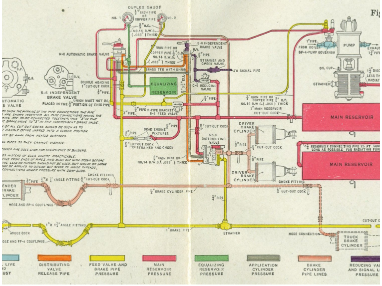
[1] Collaboration with A. Aljarbooh, A. Benveniste, T. Bourke et M. Pouzet.

Example: the Westinghouse brake





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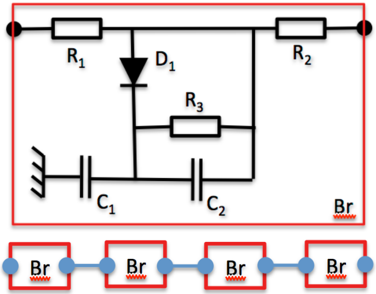
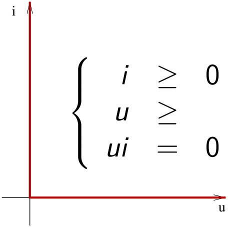


$$\left\{\begin{array}{rcl} N\mathbf{i} & = & \mathbf{0} \\ K\mathbf{u} & = & \mathbf{0} \\ \mathbf{u} - R\mathbf{i} & = & \mathbf{0} \\ C\dot{\mathbf{u}} - \mathbf{i} & = & \mathbf{0} \\ \forall i \quad 0 \leq \mathbf{i}_{D_i} & \perp & \mathbf{u}_{D_i} \geq 0 \end{array}\right.$$



$$\left\{\begin{array}{rcl} i & \geq & 0 \\ u & \geq & \\ ui & = & 0 \end{array}\right.$$

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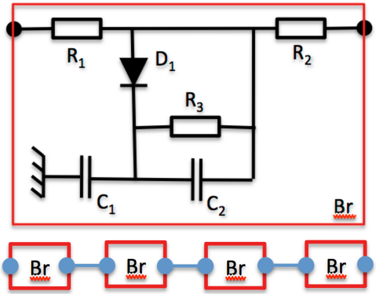

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\end{array}
\right.
$$

- 2^n modes
- Orientation and scheduling are mode dependent
- Dymola/OpenModelica : need to introduce leakage and impedance in diodes

Nonsmooth dynamical systems

Dynamical system:

$$\dot{q} = A.q + r$$

Nonsmooth perturbation r, solution of (for instance) a linear complementarity problem (LCP) :

$$\left\{ \begin{array}{rcl} r & = & B.x \\ y & = & N.q + M.x \\ 0 \leq x & \perp & y \geq 0 \end{array} \right.$$

Other nonsmooth optimisation problems: MLCP, NLCP, QP

Siconos numerical library [1, 2]

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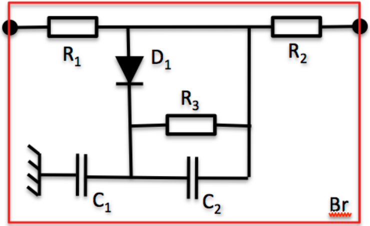
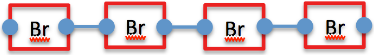
Siconos numerical library [1, 2]





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Siconos

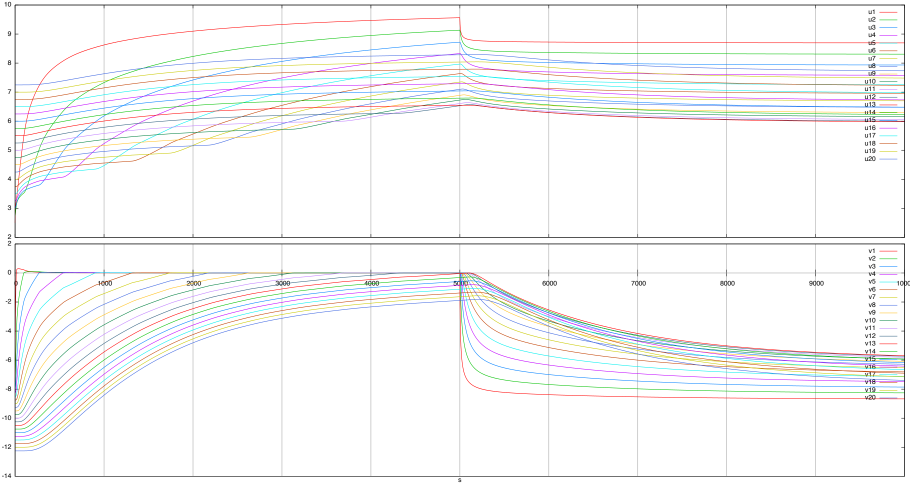
A software for modeling and simulation of nonsmooth dynamical systems

- C++ numerical library
- Several layers:
  - Kernel: nonsmooth optimization, LCP, MLCP (relay), NLCP, Newton impact law, QP
  - Numerics: system structure, discretization schemes, (N)LCS, MDI
  - Mechanics: high level multibody mechanics
- `http://siconos.gforge.inria.fr`

The Westinghouse brake in Siconos

```
#include "SiconosKernel.hpp"

int main(int argc, char * argv[])
{
  ...; // Define constants and initial state

  // --- Model definition ---

  SP::DynamicalSystemsSet NSDS(new DynamicalSystemsSet());

  // NSDS = linear time-invariant dynamical system (LTIDS) +
  //        linear complementarity problem (LCP)

  // LTIDS: q' = A.q + b + r
  SP::SiconosVector q0(new SiconosVector(2*n));
  SP::SimpleMatrix A(new SimpleMatrix(2*n,2*n));
  SP::SiconosVector b(new SiconosVector(2*n));
  ...; // Initializing q0, A and b
  SP::FirstOrderLinearTIDS RC(new FirstOrderLinearTIDS(q0,A,b));

  // LCP = linear time-inv. relation + complementarity cond.
  // y = N.q + M.x and r = B.x and 0 <= y ortho x >= 0
  SP::SimpleMatrix B(new SimpleMatrix(2*n,n));
  SP::SimpleMatrix M(new SimpleMatrix(n,n));
  SP::SimpleMatrix N(new SimpleMatrix(n,2*n));
  ...; // Initializing B, M and N
  SP::FirstOrderLinearTIR LTIR(new FirstOrderLinearTIR(*N,*B));
  LTIR->setDPtr(M);

  SP::NonSmoothLaw CC(new ComplementarityConditionNSL(n));

  // --- Connecting pieces ---

  NSDS->insert(RC);
  SP::Interaction Inter(new Interaction("I",*NSDS,1,n,CC,LTIR));
  SP::Model Circ(new Model(t0,T,Modeltitle));
  Circ->nonSmoothDynamicalSystem()->insertDynamicalSystem(RC);
  Circ->nonSmoothDynamicalSystem()->link(Inter,RC);

  // --- Initialize simulation ---

  SP::Moreau OSI(new Moreau(RC,theta)); // Moreau sweeping
  SP::TimeDiscretisation TiDisc(new TimeDiscretisation(t0,tau));
  SP::LCP OSNSP(new LCP()); // Select LCP solver
  SP::TimeStepping Strat(new TimeStepping(TiDisc,OSI,OSNSP));
  Circ->initialize(Strat); // Initialize scheme

  // --- Simulation ---

  for(k = 1 ; k < Ns ; ++k) // Time stepping
    {
      ...; // Get input
      Strat->computeOneStep(); // Solve
      Strat->nextStep(); // Transfer curr. state into last state
    }
}
```

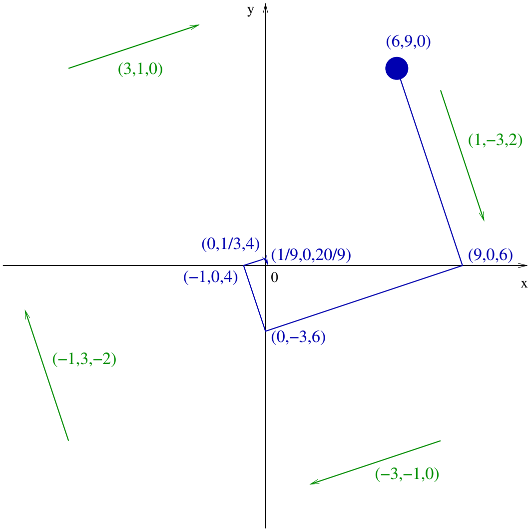
The Westinghouse brake in Siconos



Filippov differential inclusions: Zeno + sliding modes

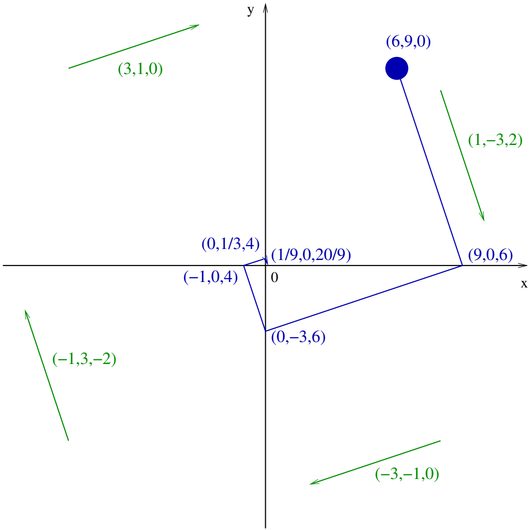
$$\begin{cases} \dot{x} & = & -\mathrm{sgn}(x) + 2\mathrm{sgn}(y) \\ \dot{y} & = & -2\mathrm{sgn}(x) - \mathrm{sgn}(y) \\ \dot{z} & = & \mathrm{sgn}(x) + \mathrm{sgn}(y) \end{cases}$$

Filippov differential inclusions: Zeno + sliding modes



$$
\begin{cases}
\dot{x} &= -\mathrm{sgn}(x) + 2\mathrm{sgn}(y) \\
\dot{y} &= -2\mathrm{sgn}(x) - \mathrm{sgn}(y) \\
\dot{z} &= \mathrm{sgn}(x) + \mathrm{sgn}(y)
\end{cases}
$$

Filippov differential inclusions: Zeno + sliding modes



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- Square "spiral"
- Attractor : line $x = y = 0$
- Zeno à $t = 7.5$
- Compute z at $t = 10$?

Filippov differential inclusions: Zeno + sliding modes

Filippov differential inclusions:

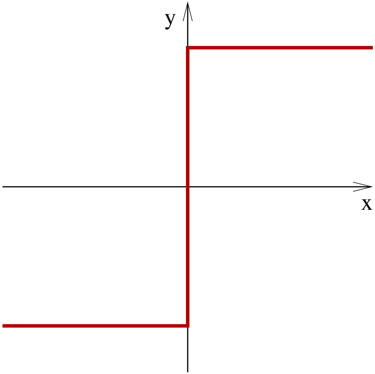
$$\dot{x} = A.x + B.\sigma$$

with:

$$\sigma \in \mathrm{sgn}(C.x)$$

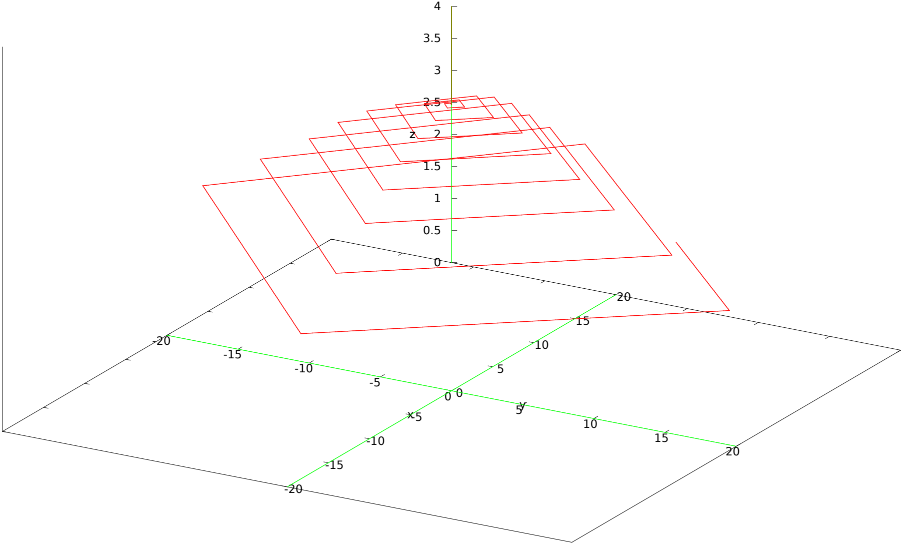
reduction to a Mixed-LCP. Averaging σ over discretization step [3]:

$$\begin{cases} M.z + q = w - v \\ -1 \leq z \leq 1 \\ (1+z).w = 0 \\ (1-z).v = 0 \\ w, v \geq 0 \end{cases}$$



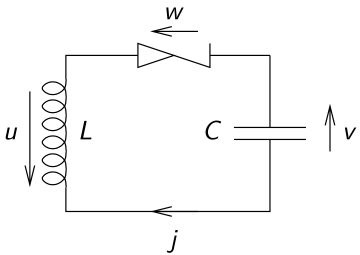
$$\sigma \in \mathrm{sgn}(y)$$
$$\Longleftrightarrow$$
$$y \in N_{[-1,1]}(\sigma) =$$
$$\{y | \forall \alpha \in [-1, 1], y(\sigma - \alpha) \geq 0\}$$

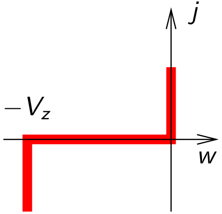
Filippov differential inclusions: Zeno + sliding modes



Detailed example: LZC

$$\begin{cases} j' &= (1/L)u \\ v' &= (1/C)j \\ 0 &= u + v + w \\ j &= j_1 - j_2 \\ w_1 &= -w \\ w_2 &= w + V_z \\ 0 \le j_1 &\perp w_1 \ge 0 \\ 0 \le j_2 &\perp w_2 \ge 0 \end{cases}$$





Detailed example: LZC

Put system into the following form:

$$\begin{cases} q' = Aq + r \\ r = Bx \\ y = Nq + Mx + d \\ 0 \leq x \perp y \geq 0 \end{cases}$$

Define:

$$q = \begin{bmatrix} j \\ v \end{bmatrix} \qquad x = \begin{bmatrix} w_1 \\ j_2 \end{bmatrix} \qquad y = \begin{bmatrix} j_1 \\ w_2 \end{bmatrix}$$

This gives:

$$A = \begin{bmatrix} 0 & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix} \qquad B = \begin{bmatrix} \frac{1}{L} & 0 \\ 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \qquad d = \begin{bmatrix} 0 \\ V_z \end{bmatrix}$$

Detailed example: LZC

Discretization: implicit Euler

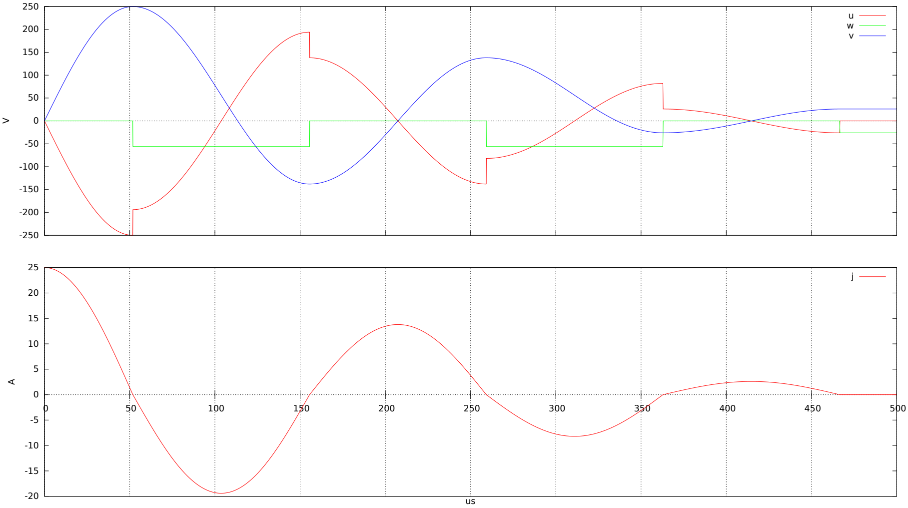
$$\frac{1}{h}(q_{n+1} - q_n) \approx Aq_{n+1} + r_{n+1}$$

With:

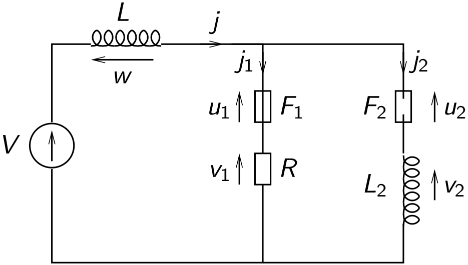
$$\begin{cases} r_{n+1} = Bx_{n+1} \\ y_{n+1} = Nq_{n+1} + Mx_{n+1} + d \\ 0 \leq x_{n+1} \perp y_{n+1} \geq 0 \end{cases}$$

Solve LCP at each step (after elimination of q_{n+1} in system above)

Detailed example: LZC

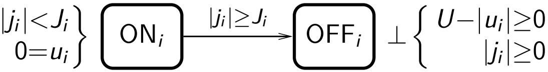


The two fuses example: Relative degree of a LCS



- Invariant:
  $j_1, j_2, u_1, u_2 \geq 0$
- Smooth dynamics:

$$\left\{ \begin{array}{rcl} j' & = & \frac{1}{L} w \\ j_2' & = & \frac{1}{L_2} v_2 \\ j_1 & = & j - j_2 \\ v_1 & = & R_1 j_1 \\ u_1 + v_1 & = & u_2 + v_2 \\ & = & V - w \end{array} \right.$$



$$\left\{ \begin{array}{l} 0 \leq j_1 \quad \perp \quad z_1 - u_1 \geq 0 \\ 0 \leq j_2 \quad \perp \quad z_2 - u_2 \geq 0 \\ \text{when } j_1 \geq J_1 \text{ do } z_1 := U_1 \\ \text{when } j_2 \geq J_2 \text{ do } z_2 := U_2 \end{array} \right.$$

The two fuses example: Relative degree of a LCS

- Put system in LCS form:

$$\begin{cases} \dot{x} & = & Ax + b + r \\ r & = & B\lambda \\ y & = & Cx + D\lambda + e \\ & & 0 \leq y \perp \lambda \geq 0 \end{cases}$$

- Choose:

$$x = \begin{bmatrix} j \\ j_2 \end{bmatrix}$$
$$y = \begin{bmatrix} j_1 \\ j_2 \end{bmatrix}$$
$$\lambda = \begin{bmatrix} z_1 - u_1 \\ z_2 - u_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

- Difficulty: $D = 0$
- LCS is ill-defined: LCP does not have a unique solution
- Derivation of the LCP gives:

$$\begin{cases} \dot{y} = CAx + Cb + CB\lambda \\ y = 0 \ \Rightarrow \ 0 \leq \dot{y} \perp \lambda \geq 0 \\ y > 0 \ \Rightarrow \ \lambda = 0 \end{cases}$$

- Relative degree $= 1$
- CB is non-singular: derived NS law has a unique solution

Conclusion

- Need to design modeling languages with nonsmooth dynamics during continuous-time evolution
- Paradigm shift:

  "Hybrid = Continuous + Discrete" to
  "Hybrid = Nonsmooth + Discrete"
- Time-stepping methods enable simulation:
  - past Zeno points
  - chattering-free sliding modes
  - Main drawback: slow convergence $O(h)$, unlike high order numerical schemes
  - Mixed adaptive event-driven / time-stepping methods
- Solutions of Filippov differential inclusions and (N)LCS are:
  - absolutely continuous
  - almost everywhere differentiable
- zero-Xing detection is as usual
- Relative degree of NSDS $\approx$ DAE index
- Not developed in this talk: Measure differential inclusions (required for Newton impact law)

References

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